Spacecraft mission design for Sun-Earth collinear libration point orbits

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The general solution to the three-body problem definitely has its difficulties. However, adding some restrictions to simplify the problem allow for modern solutions that provide stable orbits for artificial satellites placed around the equilibrium points of the reduced system. The formal definition for the equilibrium positions in the Sun-Earth/Moon system is fairly easy to derive. Based on this derivation, I will highlight a few satellite orbits that are most useful to astronomy, and pose a discussion of missions to these equilibrium points.

Keywords: Lagrange points, halo orbit, Earth-Sun system, station-keeping, dynamical systems, spacecraft, restricted three-body problem

I. INTRODUCTION

The James Webb Space Telescope (JWST) is about to launch us into the next generation of astronomical research and aerospace engineering. Before it heads out to the L2 equilibrium point 1.5 million kilometers from Earth, it can serve as an inspirational prompt to investigate the development of scientific mission design for the Sun-Earth/Moon orbital equilibrium points over the past 25 years. A few points of interest that will be highlighted in this article are the mathematical locations of these equilibrium points II, the type of orbits spacecraft will have over the course of their mission, and how the scientific objectives constrain the final mission and orbit design III.

First of all, it is important to understand the classical dynamics of these systems, even if you do not fully understand every aspect or mathematical tool used to determine the possible orbits. The system we are evaluating is the Sun-Earth/Moon-spacecraft system. The closest analytical solution possible is the circular reduced three-body problem. This method drastically reduces the complexity of the system to a point where classical, Lagrangian mechanics are able to solve for the points of equilibrium within the system. The orbits predicted from this simplified model are a great starting point for choosing the actual orbit of your spacecraft—which is usually a small perturbation from your target orbit III A 1. Station-keeping maneuvers to maintain the orbit are usually infrequent and economical from a propulsion budget perspective III A 2. Figures 2 and 3 will help you visualize these orbits.

The "equilibrium points," "libration points," or "Lagrange points," targeted for scientific missions are just two of the five total points. These two are the closest to Earth, and have been of interest for a few reasons. The most relevant advantages for the JWST mission are (1) the extremely cold and controlled environment far from Earth, and (2) the ease of maintaining the orbit over its decade-long mission IV. This reasoning applies to all the scientific missions ever sent to these points points; from the first mission, ISEE-3 in 1985, through all future proposed missions. There have already been over a dozen missions, with a lot of wisdom gained along the way. It feels like this is all culminating in the outrageously ambitious engineering, trajectory, deployment, and imagination that has been developed to bringing JWST into reality.

II. SOLVING THE CIRCULAR RESTRICTED THREE-BODY PROBLEM

A. The general vs restricted three-body problem

Imagine a system of three celestial bodies nearby each other in space. They are each exerting a gravitational force on the others. Give each of them an initial velocity, and watch what happens next. Surprisingly, there is no closed-form analytical solution to such a system. This system exhibits chaotic behavior: where small changes to initial conditions result in large changes to the outcomes after you let it evolve for a reasonable amount of time.

The restricted three body problem makes the simple restriction that the mass of the third body be negligible compared to the other two. Now, this drastically changes the problem. Instead of three independently moving bodies, you have two primary bodies, m1 and m2, and a dynamically separated, “massless” particle.

The primary bodies now form a two-body central force problem, which is fairly easy to solve using the right coordinates and assuming circular orbits. The solution of this problem will be needed to set up the next section where we will find the motion of the particle within the gravitational field of the primaries. There is no central potential for the particle's motion, so use of polar coordinates would be undesirable. Therefore, it is best to solve this with Cartesian coordinates now, so we keep everything consistent throughout the derivation.

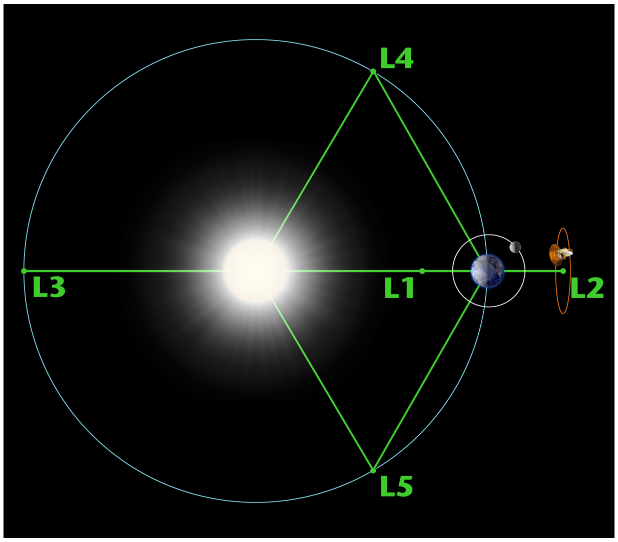

FIG. 1. The circular restricted three-body problem is shown here with the collinear points L1, L2, L3 labeled on the horizontal axis. The points L1 and L2 are of interest for their sweet-spot distance from Earth: not too far that communication becomes difficult, but not so close as to have a complex orbit of the Earth that reduces observational efficiency. In this image, the WMAP satellite is shown positioned at L2 in a Lissajous orbit. [1]

B. Gravitational Potential Two-Body Problem

1. Coordinates

The two-body problem (2BP). The coordinate system is Cartesian and rotates with the bodies. This means the entire system is confined to a 2-D plane. To simplify the position, the primaries are placed in the fixed positions [2]

$$\mathbf{r}_1 = (\mu, 0), \mathbf{r}_2 = (\mu - 1, 0), \tag{1}$$

$$|\mathbf{r}_1| = r_1 = \mu, |\mathbf{r}_2| = r_2 = 1 - \mu \tag{2}$$

The vector $\mathbf{r}$ separating them is then

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (1, 0) \tag{3}$$

The masses are fixed at

$$m_1 = 1 - \mu, m_2 = \mu, \mu \in [0, 0.5] \tag{4}$$

2. Newton's method

Now that we have the coordinates set up, we can look at the forces. From Newtonian physics, we can see that the gravity and centrifugal forces must be balanced to maintain a stable orbit.

$$\frac{Gm_1m_2}{r^2} = m_1\omega^2 r_1 = m_2\omega^2 r_2 \tag{5}$$

$$\frac{G\mu(1-\mu)}{1^2} = (1-\mu)\mu\omega^2 = \mu(1-\mu)\omega^2 \tag{6}$$

$$G\mu(1-\mu) = \omega^2\mu(1-\mu) = \omega^2\mu(1-\mu) \tag{7}$$

$$G = \omega^2 \tag{8}$$

Our choice of coordinates gave a quick result after balancing the forces.

3. Lagrange's method

We can also look at the Lagrangian L of the system. Converting all the forces to potential energy, you get

$$\mathcal{L} = T - U = -U \tag{9}$$

Recall

$$F = -\frac{\partial U}{\partial x}, \tag{10}$$

Therefore,

$$U = -\int F dx \tag{11}$$

$$\mathcal{L}_1 = \frac{Gm_1m_2}{r} - \frac{1}{2}m_1\omega^2 r_1^2 - \frac{1}{2}m_2\omega^2 r_2^2 \tag{12}$$

$$\mathcal{L}_1 = \frac{G\mu(1-\mu)}{1} - \frac{1}{2}\omega^2(1-\mu)\mu^2 - \frac{1}{2}\omega^2\mu(1-\mu)^2 \tag{13}$$

Plugging in G = ω^2

$$\mathcal{L}_1 = G\mu(1-\mu)[1 - \frac{1}{2}\mu - \frac{1}{2}(1-\mu)] \tag{14}$$

$$\mathcal{L}_1 = G\mu(1-\mu)[1 - \frac{1}{2} - \frac{1}{2}\mu + \frac{1}{2}\mu] \tag{15}$$

$$\mathcal{L}_1 = \frac{1}{2}G\mu(1-\mu) \tag{16}$$

Voila!

C. Particle in a two-body gravitational field

Now that we know the Lagrangian of the two-body system, we can add in the contribution of the particle to find the Lagrangian of the three-body system.

1. Coordinates

The position of the particle is

$$\mathbf{R} = (x, y), |\mathbf{R}| = R = \sqrt{x^2 + y^2} \tag{17}$$

The distances from the particle to bodies one and two are, respectively,

$$R_1 = |\mathbf{R} - \mathbf{r}_1| = \sqrt{(x-\mu)^2 + y^2} \tag{18}$$

$$R_2 = |\mathbf{R} - \mathbf{r}_2| = \sqrt{(x-\mu+1)^2 + y^2} \tag{19}$$

and remember the mass is negligible.

2. Contribution to the Lagrangian

Repeat the process. Notice this time there is one centrifugal term, and two gravitational terms.

$$\mathcal{L}_2 = \frac{Gm_1}{R_1} + \frac{Gm_2}{R_2} - \frac{1}{2}\omega^2 R^2, \tag{20}$$

$$\mathcal{L}_2 = G(\frac{(1-\mu)}{R_1} + \frac{\mu}{R_2} - \frac{1}{2}(x^2+y^2)). \tag{21}$$

So, adding all the energies together you get a Lagrangian for the whole system

$$\mathcal{L} = \frac{1}{2}G\mu(1-\mu) + \frac{G(1-\mu)}{R_1} + \frac{G\mu}{R_2} - \frac{1}{2}G(x^2+y^2), \tag{22}$$

$$\mathcal{L} = G(\frac{1}{2}\mu(1-\mu) + \frac{(1-\mu)}{R_1} + \frac{\mu}{R_2} - \frac{1}{2}(x^2+y^2)). \tag{23}$$

D. Equations of Motion

Plugging into the Euler-Lagrange equations for x,

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \tag{24}$$

$$G\frac{\partial}{\partial x}(\frac{1-\mu}{\sqrt{(x-\mu)^2+y^2}} + \frac{\mu}{\sqrt{(x-\mu+1)^2+y^2}} - \frac{1}{2}(x^2+y^2)) = 0 \tag{25}$$

$$\frac{\partial}{\partial x}\frac{1-\mu}{\sqrt{(x-\mu)^2+y^2}} + \frac{\partial}{\partial x}\frac{\mu}{\sqrt{(x-\mu+1)^2+y^2}} - \frac{\partial}{\partial x}\frac{1}{2}(x^2+y^2) = 0. \tag{26}$$

Therefore, the equation of motion for x is [3]

$$\ddot{x} - 2\dot{y} = \frac{\partial \mathcal{L}}{\partial x} = 0. \tag{27}$$

Repeat for y. The equation of motion for y is

$$\ddot{y} + 2\dot{x} = \frac{\partial \mathcal{L}}{\partial y} = 0. \tag{28}$$

E. Jacobian integral and Jacobi constant

If we divide L by G, we can define a new equation Ω

$$\frac{\mathcal{L}}{G} \equiv \Omega \tag{29}$$

The Jacobian integral and Jacobi constant C [2] for Ω are

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C \tag{30}$$

$$C = 2\Omega - \dot{x}^2 - \dot{y}^2 \tag{31}$$

F. Solving the equations of motion at the collinear libration points

Collinear means they all lie along the x-axis, and therefore have y=0. The positions are then

$$L_2 < \mu - 1 < L_1 < \mu < L_3 \tag{32}$$

Rewriting Ω to get the Hamiltonian H now,

$$\dot{x} - y = p_x \tag{33}$$

$$\dot{y} + x = p_y \tag{34}$$

$$H = \frac{1}{2}(p_x^2 + p_y^2) - xp_y + yp_x - \frac{1-\mu}{R_1} - \frac{\mu}{R_2} \tag{35}$$

Hamilton's equations give

$$\dot{x} = p_x + y \tag{36}$$

$$\dot{p_x} = p_y - \frac{(1-\mu)(x-\mu)}{R_1^3} - \frac{\mu(x-\mu+1)}{R_2^3} \tag{37}$$

$$\dot{y} = p_y - x \tag{38}$$

$$\dot{p_y} = -p_x - \frac{(1-\mu)y}{R_1^3} - \frac{\mu y}{R_2^3} \tag{39}$$

Now we can rewrite the Jacobi constant C

$$C = -2H - \mu(1-\mu) \tag{40}$$

> "The linearized equations around any collinear equilibrium point are given by the second-order terms of the Hamiltonian H2. The characteristic polynomial associated to the planar motion is p(lam) [2]."

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) - \dot{x}p_y + \dot{y}p_x - c_2(x^2 - \frac{1}{2}y^2) \tag{41}$$

$$c_2 > 1 \tag{42}$$

$$p(\lambda) = \lambda^4 + (2-c_2)\lambda^2 + (1 + c_2 - 2c_2^2) \tag{43}$$

Replacing λ

$$\lambda^2 \rightarrow \eta \tag{44}$$

$$\eta = \frac{c_2 - 2 \pm \sqrt{9c_2^2 - 8c_2)}}{2} \tag{45}$$

$$\eta_1 > 0 \tag{46}$$

$$\eta_2 < 0 \tag{47}$$

> "This result tells us the three equilibrium points are of the type center x center x saddle [2]."

Now that we have determined the nature of and behavior around the libration points, the next step is to design a trajectory and orbit that will let you stay nearby, with only small propulsion mass expenditures.

III. ORBITING THE COLLINEAR LIBRATION POINTS

The math that describes the families of periodic and quasi-periodic orbits is beyond the scope of this paper. Instead, we will cover an overview of their qualitative features.

A. Families of orbits

The families of periodic and quasi-periodic orbits are equally important for mission trajectory design. Periodic Lyapunov and halo orbits make up the first part of orbit design, while quasi-periodic Lissajous orbits make up the latter portion of the design. The mission has to be designed with periodic (not-chaotic) orbits because these are actually solvable. They will be modeled analytically alongside known mission parameters: the form of the spacecraft, the scientific mission objectives, fuel limitations, etc. Once the best orbit has been identified, this is set as the target orbit for the mission. The next step will hold that as the target while the model is modified to become more realistic. Introducing chaos-inducing perturbations is the best preparation for real situations that will present themselves in orbit [4][5]. You can only stay in a quasi-periodic, stable orbit if you maintain it via station-keeping with the use of thrusters from time to time. If you let enough time pass, you would exponentially drift from the unstable equilibrium, or "saddle" point. To prevent anomalies and reduce risk, physicists use Monte Carlo simulations to determine a conservative station-keeping fuel budget. [6]

Recently, articles describing orbit design for certain classes of yet-unannounced missions are starting to emerge. The calculations are made to be easily tailored to a specific scenario if it is within the scope of the paper. Although these scientists are not planning for any mission directly, they are definitely setting the stage for more libration point missions. One great example is a recent article on orbital trajectories of solar-sail spacecraft [7].

1. *Periodic orbits*

The periodic orbit families relevant to modern missions include the planar Lyapunov, vertical Lyapunov, and halo families. Periodic orbits appear constant from all points of view [8]. Halo orbits are three-dimensional orbit whereas Lyapunov orbits are two-dimensional and lie in the plane of the primary bodies or perpendicular to it. Halo orbits tend to be the largest. This is an advantage because less orbital maintenance has to be done, and therefor fuel-costs are lowered and mission lifetime is extended [8]. See Fig.2 from [5] for an example of halo orbits calculated in 1977 for the 1985 International Sun-Earth Explorer (ISEE-3) mission.

2. *Quasi-periodic orbits*

Quasi-periodic orbits are more challenging because they must be calculated with numerical methods instead of the preferable analytical ones. Lissajous orbits are the only quasi-periodic, semi-stable orbits that a spacecraft can maintain at L1 and L2. These orbits actually require less station-keeping than periodic ones because they are closer to the natural motion of a body at a libration point. The plane of the orbit precesses perpendicular to the plane of the primary bodies. This means an eclipse is inevitable without a maneuver to avoid it see FIG.4.

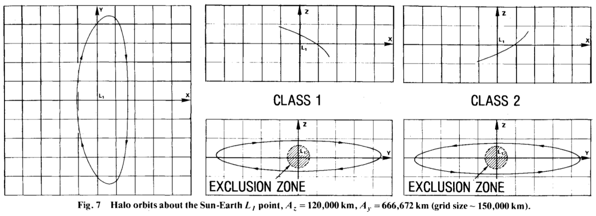

FIG. 2. Simulated periodic orbits for a spacecraft around the collinear libration point, L1, of the Sun-Earth/Moon System. These "halo orbits" are stable and very well understood. In the simulation, they are fairly simple, but they are not practical in real life. Scientists must figure out how to accurately model the effects of complex dynamics of the solar system on the spacecraft in orbit to refine these orbits into even more useful models [5].

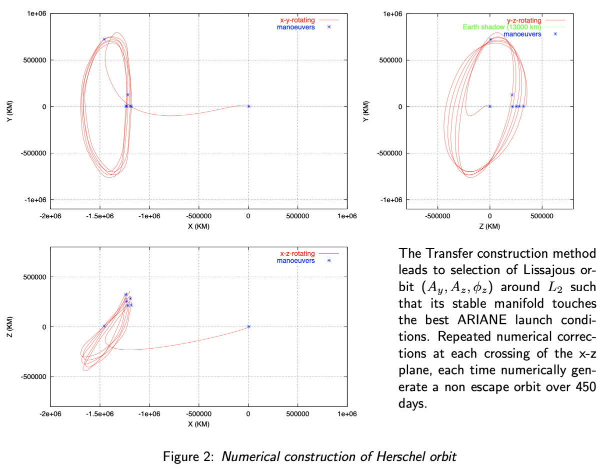

FIG. 3. Simulated quasi-periodic orbits for the Herschel spacecraft around the collinear libration point, L2, of the Sun-Earth/Moon System. These Lissajous orbits are unstable and hard to model, as they are sensitive to initial conditions. They do resemble real-life more than periodic orbits, so they are scientifically very useful in designing the orbit of a spacecraft [10]

This is a major drawback of these orbits, as some missions simply could not withstand a loss of power for the duration of the eclipse. Luckily it would be feasible to have an orbit reversal at the point just before an eclipse. This "time-back" method would only have to be done every 6 years at L2 to completely avoid eclipses [9]. See FIG. 3 for an example of stage-two orbit design for the Herschel spacecraft.

B. Selected applications of orbit design

The history of these orbits goes back to the 1960s when Robert Farquhar pioneered the field of spacecraft trajectory at the beginning of the Space Race. He designed most of NASA's spacecraft orbits from the 1980s through the mid-2000s [10]. Since then, there has been widespread use of all of these types of orbits as they are able to be tailored to the exact needs of the mission.

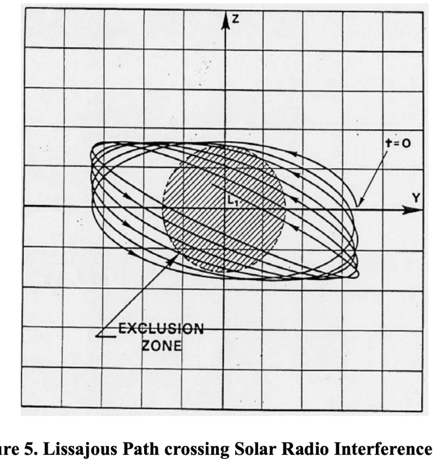

FIG. 4. This figure shows the path of a Lissajous orbit at L1 whose trajectory puts it in an area of zero communication while in the Solar Radio Interference Zone [10].

Although many missions are currently being planned, the orbits they each have designed are very diverse. Certain orbits that would be ideal for one spacecraft would be extremely inefficient and unstable for another. For example, a spacecraft with a small surface area pointing toward the sun would have a small perturbation from the pressure of the solar wind. Whereas a spacecraft with a tennis court sized Sun-shield meant to bounce back everything coming from the Sun would act as a solar-sail and have a much larger perturbation to have to account for. This is the case for JWST, which has a much more frequent station-keeping schedule than if it was not wearing a giant sail on its back. The station-keeping maneuvers increased in frequency from a baseline of 3-4 times per year to every 21 days [11]. Things get more complicated as you dive deeper, as there is both a force and a torque on the spacecraft from the pressure from the solar wind. Orbital station-keeping maneuvers must be designed to off-load these extra forces in an efficient way [11].

IV. DISCUSSION: COMPLEXITIES OF SCIENTIFIC MISSION DESIGN FOR LIBRATION POINT TRAJECTORIES

There is a seemingly endless list of things to consider when designing a scientific spacecraft for the libration points. Here are a few that I have gethered are quite important:

- efficiency: fuel - how often the spacecraft needs to correct orbit, observational - how much of the mission time is spent observing [8],

- risk management: margin of error to get maximum benefit from orbital trajectory without risking an eclipse [9] [10],
- usefulness: how exactly do you quantify how much a spacecraft benefits from a certain orbit at a certain libration point [2],
- reasoning: what types of scientific missions would an orbit be best suited for and can or should we get public support for missions of that type [12].

Luckily there is a main focus for these extraterrestrial observatories. They are usually doing science that is not possible on or nearby the Earth: [10].

- ISEE-3: observed the full spectrum of the Earth's geomagnetic tail,
- ACE, Wind, Genesis: study the solar wind before it reaches Earth,
- WMAP, Planck: need very cold temperatures to study cosmic microwave background,
- Herschel, JWST, NGST: also need cold temperatures to study deep space and infrared, also need extreme precision.

V. CONCLUSIONS

This was an undergraduate-level overview of the classical, Lagrangian mechanics of the collinear libration points of the circular restricted three-body problem. We saw how the points along the collinear axis through the two primary bodies lead to unstable, saddle equilibrium points.

We took that concrete knowledge and applied it to the much more technically difficult idea of the possible orbits spacecraft could actually maintain around these points. This uncovered the truly unpredictable nature of the three-body problem and the countless chaotic solutions it has. We thought about some of the main mission objectives and dynamical factors that would affect the design of the orbit.

After all this consideration of perturbations and objectives, we finished with a broad, open-ended discussion of the challenges and applications of these orbits and the scope of scientific missions to the libration points.

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